A shortcut to sign Incremental Value-at-Risk for risk allocation[*]

Dirk Tasche[†]   Luisa Tibiletti[‡]

October 2, 2002

Abstract

Approximate Incremental Value-at-Risk formulae provide an easy-to-use preliminary guideline for risk allocation. Both the cases of risk adding and risk pooling are examined and beta-based formulae achieved. Results highlight how much the conditions for adding new risky positions are stronger than those required for risk pooling.

JEL CLASSIFICATION: *C13; D81; G11; G12.*

KEY WORDS: *Incremental Value-at-Risk (IVaR); Risk pooling; Risk adding.*

---

1 Introduction

Incremental Value-at-Risk (IVaR) is becoming a standard tool in investment management industry to identify strategies that enhance returns and control risk. In theory, IVaR is a metrics that measures the contribution in terms of relative Value-at-Risk (VaR) of a position or a group of positions with respect to the total risk of a pre-existent portfolio. In the academic literature the attention to this technique can be traced back to the some works by Dowd [1998, 1999, 2000]. Nevertheless, in recent times more practice-oriented researchers have dwelt on this tool for a twofold purpose: 1) hedging and speculating with options (see Mina [2002]) and 2) reducing the risk in a risk-return analysis. Two slightly different approaches follow: one leads to a risk adding model, the other to a risk pooling model.

Risk adding means increasing the amount of invested money by purchasing a new asset. This new asset will be a hedge if the portfolio risk is reduced by its addition. When risk is measured with VaR, risk reduction is indicated by a negative IVaR.

Pooling risks means to keep the amount of invested money constant. Hence, pooling a portfolio with a new asset forces reduction of the weights of the extant assets. In this case also a negative IVaR indicates reduction of risk. However, we will see below that the set of assets that reduce risk when pooled with a portfolio is much larger than the set of hedges.

In theory, a straightforward way to calculate IVaR requires us to create a new portfolio incorporating the candidate new strategy, then reassess VaR and finally compare it with the previous one. Although in recent time, techniques for upgrading the construction of the portfolio probability distributions have been proposed (see Wang [2002]), due to the non-linearity of VaR, computation may still turn out to be too time-expensive thus being a bar to real-time decision making. Having at disposal friendly-to-use approximating formulae may be a first step to overcome this drawback. This is just the aim of the paper.

Beta-based approximations of IVaR are achieved. Formulae highlight a too often underestimated aspect. By virtue of the different diversification impact, conditions for adding new positions in the portfolio are much stronger than those for their pooling. A final warning. Although the formulae are more reliable for elliptical returns, as we skip behind this assumption they might grow lower in confidence, since they provide only a linear approximation. Clearly, higher order approximations should be worked out. Although this is technically attainable, it drives to cumbersome formulae which will be in contrast to the spirit of this paper. In conclusion, the linear approximation formulae can be a useful tool for a preliminary screening of the alternatives in risk allocation.

The paper is organized as follows. In Section 2 Incremental VaR (IVaR) is defined. An approximate IVaR formula for risk adding is introduced in Section 3, whereas one for risk pooling occurs in Section 4. A conclusion in Section 5 ends the note.

2   Incremental VaR

VaR measures the smallest level of under-performance of a position that would occur with a low probability by a given time horizon. In the following, we will deal with the *excess return* of a position with reference to a benchmark. The benchmark may vary with the context: it could be the liabilities for pension funds, the investment benchmark for traditional asset managers, or just the cash for hedge funds. Let us give a sketchy definition list. Define for random a variable Z and $\alpha \in (0,1)$ the α-quantile q_α of Z by

$$q_\alpha(Z) \quad \overset{\text{def}}{=} \quad \inf\{z \in \mathbb{R} \mid \mathrm{P}[Z \leq z] \geq \alpha\}. \tag{2.1}$$

In what follows, we use the definition of VaR *relative to the mean* in the sense[1] of Jorion [1997, page 87]:

$$\mathrm{VaR}_\alpha(Z) \quad \overset{\text{def}}{=} \quad \mathrm{E}(Z) - q_\alpha(Z) \tag{2.2}$$

Let us focus on the impact on the current portfolio by a prospective asset purchase or disposal. Stand-alone the risk involved in the individual asset, we need also to take into account the effects of this position on the new aggregate portfolio. For notation convenience, let us fix α (usually $\alpha = 0.01$ or $\alpha = 0.05$) and drop the symbol α in future formulae. Thus, we define the *Incremental VaR (IVaR)* as the difference between the VaR of the new and the current portfolio:

$$\text{Incremental VaR} = \mathrm{VaR}\,(\text{new portfolio}) - \mathrm{VaR}\,(\text{current portfolio})$$

Clearly, IVaR can be positive, if the candidate strategy adds risk to the current portfolio, or negative, if the strategy acts like a hedge to the existing portfolio risks, or zero if it is neutral.

In the sequel, we will deal with two strategies to allocate risk, termed *risk adding* and *risk pooling.*

---

[1] Another possibility would be to use the *absolute* definition $\mathrm{VaR}_\alpha(Z) \overset{def}{=} -q_\alpha(Z)$. Clearly, the two of definitions coincide if $\mathrm{E}(Z) = 0$.

3   IVaR for risk adding

Let X denote the random excess return of the current portfolio. Suppose we want to measure the contribution of a position Y or a group of positions to the total risk of the portfolio. Two situations may occur. We could want to measure the portfolio risk after having sold Y or a portion aY of it, and having invested the proceeds in cash. But we could also focus on the portfolio after having bought the position or a portion aY of it, by drawing the requested money from cash. In any case, the result is

$$\text{new portfolio} \quad \overset{\text{def}}{=} \quad X + aY \tag{3.1}$$

where $a < 0$ and $a > 0$ refer to the selling and buying case, respectively; clearly, if $a = 0$ no change in the portfolio occurs.

According to the definition of IVaR, we get

$$\text{IVaR} = \text{VaR}\,(X + aY) - \text{VaR}\,(X)$$

Writing IVaR as a function of the variable a, we obtain

$$f\,(a) = \text{VaR}\,(X + aY) - \text{VaR}\,(X) = a\,\text{E}\,[Y] + q(X) - q\,(X + aY)$$

An easy way to get a rough estimate of IVaR is to look for a linear approximation. Since $f\,(0) = 0$, in case of f being differentiable with respect to a the sign of IVaR for small positive a is just the sign of $f'(0)$. Conditions for f to be differentiable and an explicit formula for the derivative are provided in Gouriéroux at al. [2000], Lemus [1999], and Tasche [1999]. Application of this formula yields

$$f'\,(a) = \text{E}\,[Y] - \text{E}\,[Y|X + aY = q\,(X + aY)],$$

where $\text{E}[Y|Z = z]$ denote the conditional expectation of Y given that the random variable Z equals z, and in particular

$$f'\,(0) = \text{E}\,[Y] - \text{E}\,[Y|X = q\,(X)].$$

Observe that $\text{E}\,[Y|X = q\,(X)] - \text{E}\,[Y]$ is the best prediction of $Y - \text{E}\,[Y]$ given that $X - \text{E}\,[X] = q\,(X) - \text{E}\,[X]$. In the case of the conditional expectation $\text{E}\,[Y|X = \cdot]$ not

4

being available, a reasonable approximation would be the best *linear* prediction of $Y - \mathrm{E}[Y]$ given that $X - \mathrm{E}[X] = q(X) - \mathrm{E}[X]$. This linear prediction is given by

$$(q(X) - \mathrm{E}[X]) \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)}, \tag{3.2}$$

where cov and var denote covariance and variance respectively, as usual. By (2.2), we therefore obtain $f'(0) \approx \beta_{yx} \mathrm{VaR}(X)$, where $\beta_{yx} = \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)}$ is the standard beta coefficient of Y with respect to X. This way, a linear approximation of IVaR comes out as

$$IVaR \approx \beta_{yx} \mathrm{VaR}(X) \, a. \tag{3.3}$$

Now, a clear-cut tool for discriminating profitable strategies has emerged. Just a glance at the sign of β_{yx} is sufficient to give a rough information on the sign of the risk contribution of the position Y.

Remark 1   *(i) If we are considering the position aY for purchasing (so that $a > 0$), the negativeness of β_{yx} signals that it will act like a risk diversifier. Vice versa, if we are looking for measuring the risk contribution of the position aY already contained in the portfolio, the negativeness of β_{yx} signals that when sold it will act like a risk contributor. In conclusion, as intuition suggests, adding aY tends to reduce the risk only if Y is a super-defensive asset, i.e. the return goes in the opposite direction of that of X. Vice versa, selling aY tends to reduce the risk only if Y is a conservative or aggressive asset with respect to X, i.e. the return of Y goes in the same direction of that of X.*

   (ii) Approximation (3.3) holds under very loose conditions on distributions. For instance, it suffices that X and Y have a continuous joint density. This is a standard assumption in financial modeling.

   (iii) Recall that (3.3) comes out as the result of two consecutive approximations. Firstly, we have replaced $f(a)$ by the term $af'(0)$. Secondly, we have substituted (3.2) for $f'(0)$. It can be proved that the best linear prediction by (3.2) coincides with the best prediction $\mathrm{E}[Y|X = q(X)] - \mathrm{E}[Y]$ if X and Y are normal returns (or more general, if X and Y are jointly elliptical distributed[2]). This indicates that approximation (3.3) will be reliable for small a at least in a "normal" (or elliptical) world. Nevertheless, as we skip behind this assumption, the formula has to be handled with caution since it may lead to a misleading information.

---

[2]Elliptical distributions, include the normal distribution as a special case, as well as the Student's t-distribution and the Cauchy distribution. Elliptical distributions are called "elliptical" since the contours of the density are ellipsoids. Kelker [1970] proved that VaR can be written in term of standard deviation when the underlying distribution is elliptical.

(iv) One way to to improve on the approximations is working out higher order approximations (see Gouriéroux at al. [2000] for the second order derivative of VaR). But a dark side of the coin exists. The higher the approximation order is the more cumbersome the formulas are. So, this approach is just in contrast to the spirit of this paper.

4   IVaR for risk pooling

As in the previous section, let X be the random excess of return of the current portfolio. Let aY be the position we are considering to purchase. The aim in this case is just to arrive at portfolio diversification. If the asset is purchased, the portfolio is re-balanced, so

$$\text{new portfolio} \quad \overset{\text{def}}{=} \quad \frac{X + aY}{(1 + a)} \tag{4.1}$$

where $a > 0$ and the factors $\frac{1}{(1+a)}$ and $\frac{a}{(1+a)}$ are the relative weights of the assets X and Y, respectively. Therefore,

$$\text{IVaR} = \text{VaR (new re-balanced portfolio)} - \text{VaR (old portfolio)}.$$

Proceeding as in the risk adding case, let us write IVaR as a function of the variable a:

$$g\left(a\right) = \text{VaR}\left(\tfrac{X+aY}{1+a}\right) - \text{VaR}\left(X\right) = \text{E}\left[\tfrac{X+aY}{1+a}\right] - q\left(\tfrac{X+aY}{1+a}\right) - \text{E}\left[X\right] + q\left(X\right)$$

In particular, we have $g\left(0\right) = 0$. In case of g being differentiable with respect to a we obtain

$$g'\left(a\right) = \left(1 + a\right)^{-2} \text{E}\left[Y\right] - \text{E}\left[X\right] + q\left(X + aY\right) - \left(1 + a\right) \text{E}\left[Y | X + aY = q\left(X + aY\right)\right],$$

and in particular

$$g'\left(0\right) = \text{E}\left[Y\right] - \text{E}\left[Y | X = q\left(X\right)\right] - \text{VaR}\left(X\right)$$
$$\approx \left(\text{E}\left[X\right] - q\left(X\right)\right) \tfrac{\text{cov}(X,Y)}{\text{var}(X)} - \text{VaR}\left(X\right).$$

Therefore $g'\left(0\right) \approx \left(\beta_{yx} - 1\right) VaR\left(X\right)$, and the linear approximation of IVaR turns out to be

$$\text{IVaR} \approx a\left(\beta_{yx} - 1\right)\text{VaR}\left(X\right). \tag{4.2}$$

Remark 2 *(i) Although formula (4.2) looks similar to (3.3), a striking difference sticks out. In case of pooling, the watershed for discriminating profitable strategy is no longer the sign of β_{yx}, but the sign of $(\beta_{yx} - 1)$. So, it may happen that even if β_{yx} is positive (but less than one) and hence X and Y are positively correlated, pooling of at least a small portion aY may be always advisable for reducing the risk. Let us be more precise, consider the case of $0 < \beta_{yx} < 1$, i.e. Y is a defensive asset. Because of positive correlation, Y goes in the same direction as X, but Y varies less than X does. This means that the variations of the excess return Y grow lower than those of X do. In the case of $\beta_{yx} \leq 0$, i.e. Y is a super-defensive asset, variations of Y go in the opposite direction of those of X. In conclusion, except in the case of a super-aggressive asset with $\beta_{yx} > 1$, pooling at least a small portion aY is always an advisable strategy for reducing risk.*

(ii) The fact that the risk pooling conditions are much looser than those for risk adding, should not surprise. In a normative framework, the argument about different attitudes in accepting risk adding and risk pooling can be tracked back to the so-called "Samuelson's Fallacy of Large Numbers" (see Ross [1999] and the references thereby). By virtue of the favorable diversification effect, the acceptance of pooling in the portfolio a sufficiently long string of single-rejected risks is considered "rational". Vice versa, the eventual acceptance of adding single-rejected risks is much more questionable.

(iii) Again, (4.2) is more reliable for elliptically distributed returns, but may lose its reliability as we relax this assumption.

5   Conclusion

Beta-based approximation formulae for IVaR are achieved in both cases – adding a new risk and pooling it with the existing portfolio. These approximations are based on recent results on the differentiability of VaR. The formulae will give the right indications for trade decisions as long as the weights of the assets under consideration are small compared to the weights of the unchanged assets. One can see from the formulae that the conditions for adding new positions are by far stronger than those for pooling.

References

Dowd, K. "Beyond value at risk." John Wiley & Sons, 1998.

Dowd, K. "A value at risk approach to risk-return analysis." *The Journal of Portfolio Management*, 25, Summer 1999, 60–67.

Dowd, K. "Adjusting for Risk: An Improved Sharpe Ratio." *International Review of Economics and Finance*, 9, 2000, 209-222.

Gouriéroux, C., Laurent, J. P. and O. Scaillet. "Sensitivity analysis of Values at Risk." *Journal of Empirical Finance*, 7, 2000, 225-245.

Kelker, D. "Distribution theory of spherical distributions and a location-scale parameter generalization." *Sankhyā Ser. A* 32, 1970, 419–438.

Lemus, G. "Portfolio Optimization with Quantile-based Risk Measures." PhD thesis, Sloan School of Management, MIT.
`http://citeseer.nj.nec.com/lemus99portfolio.html`

Mina, Jorge. "Measuring Bets With Relative Value at Risk." *Derivatives Week*, Learning curve, January 2002, 14-15.

Jorion, P. "Value at risk: the new benchmark for controlling market risk." Irwin, Chicago 1997.

Ross, S. "Adding Risks: Samuelson's Fallacy of Large Numbers Revisited." *Journal of Financial and Quantitative Analysis*, 34(3), 1999, 323-339.

Tasche, D. "Risk contributions and performance measurement." Working paper, Technische Universität München 1999. `http://www.ma.tum.de/stat/`

Wang, Z. "The Properties of Incremental VaR in Monte Carlo Simulations." *The Journal of Risk Finance*, 3, n. 2, 2002, 14-23.